

INEQUALITIES OF EXTENDED (p, q) -BETA AND CONFLUENT HYPERGEOMETRIC FUNCTION

SHAHID MUBEEN, KOTTAKKARAN SOOPPY NISAR*,
GAUHAR RAHMAN AND MUHAMMAD ARSHAD

ABSTRACT. In this present paper, we establish the log convexity and Turán type inequalities of extended (p, q) -beta functions. Also, we present the log-convexity, the monotonicity and Turán type inequalities for extended (p, q) -confluent hypergeometric function by using the inequalities of extended (p, q) -beta functions.

1. INTRODUCTION

We begin with the classical gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

In another way, it is defined as

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^{z-1}}{(z)_n}$$

where $(\alpha)_n$ is the Pochhammer symbol defined as

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+n-1); & \text{for } n \geq 1, \alpha \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

and

$$\Gamma(z+1) = z\Gamma(z)$$

The relation between Pochhammer symbol and gamma function is given below

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

The beta function is defined by

$$\beta(x, y) = \int_0^{\infty} t^{x-1} (1-t)^{y-1} dt, \quad (1.1)$$

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*Corresponding author.

$$(\Re(x) > 0, \Re(y) > 0)$$

and

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \Re(x) > 0, \Re(y) > 0. \quad (1.2)$$

Chaudhry and Zubair [4] and Chaudhry et al. [5] defined the following extended gamma and beta functions

$$\Gamma_p(z) = \int_0^\infty t^{z-1} e^{-t-pt^{-1}} dt, \quad (1.3)$$

$\Re(z) > 0, p \geq 0$. When $p = 0$, then Γ_p tends to the classical gamma function Γ , and

$$\beta(x, y; p) = \int_0^\infty t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt \quad (1.4)$$

(where $\Re(p) > 0, \Re(x) > 0, \Re(y) > 0$) respectively. When $p = 0$, then $\beta(x, y; 0) = \beta(x, y)$. Recently Choi et al. [6] introduced the following extension of extended beta function as

$$\beta(x, y; p, q) = \beta_{p,q}(x, y) = \int_0^\infty t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t} - \frac{q}{1-t}} dt \quad (1.5)$$

(where $\Re(p) > 0, \Re(q) > 0, \Re(x) > 0, \Re(y) > 0$).

It is clear that when $p = q$, then (1.5) reduces to the well known extended beta function (1.4). Similarly if $p = q = 0$, then (1.5) reduces to the classical beta function (1.1).

In the same paper, they also defined the following extension of extended confluent hypergeometric function by

$$\Phi_{p,q}(\beta; \gamma; z) = \sum_{n=0}^\infty \frac{\beta(\beta+n; \gamma-\beta; p, q)}{\beta(\beta, \gamma-\beta)} \frac{z^n}{n!} \quad (1.6)$$

$$(p \geq 0, q \geq 0, \Re(\gamma) > \Re(\beta) > 0),$$

The integral representations of extension of extended confluent hypergeometric function is given by

$$\Phi_{p,q}(\beta; \gamma; z) = \frac{1}{\beta(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \exp\left(zt - \frac{p}{t} - \frac{q}{1-t}\right) dt \quad (1.7)$$

$$(p \geq 0, q \geq 0, \Re(\gamma) > \Re(\beta) > 0).$$

Note that for $p = q$, the series (1.6) respectively reduces to the extended confluent hypergeometric series. Similarly for $p = q = 0$ the series (1.6) respectively reduces to the classical confluent hypergeometric series.

The main objective of this paper is to establish the log-convexity of extension of extended beta and confluent hypergeometric functions. In particular, we define the Turán type inequality [13] of the above two extended functions. The plan of this paper is follows:

In section 2, we present several inequalities for extension of extended beta function also called (p, q) -beta function. In this section, the classical Chebyshev's integral inequality and the Hölder-Roger inequality for integrals are used to obtain the main results. Section 3 is devoted to the log-convexity and the Turán type inequality of extended (p, q) -confluent hypergeometric function.

2. MAIN RESULTS: INEQUALITIES OF EXTENDED (p, q) -BETA FUNCTION

In this section, we establish some inequalities which involve extended (p, q) -beta functions by using some natural inequalities [10]. For this continuation of our study, we recall the following well-known Chebychev's integral inequality and Hölder-Rogers inequality.

Lemma 2.1. (see [7, 8]) *Let the functions $f, g : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ are asynchronous for all $x \in [a, b]$ and $p(x) : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a positive integrable function, then*

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \quad (2.1)$$

Definition 2.1. *In [3], a function $f : (a, b) \rightarrow \mathbb{R}$ is said to be convex if for any $x_1, x_2 \in (a, b)$ and $\alpha \in (0, 1)$*

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2). \quad (2.2)$$

It shows that when we move from x_1 to x_2 , the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies always above the graph of f .

Definition 2.2. *A function f is said to be a log-convex if $f > 0$ and $\log f$ is convex i.e., for all $x_1, x_2 \in I$ (where I is an interval) and $\alpha \in (0, 1)$, we have*

$$\log f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha \log f(x_1) + (1 - \alpha) \log f(x_2) = \log(f^\alpha(x)f^{1-\alpha}(x_2)).$$

This implies that

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq f^\alpha(x_1)f^{1-\alpha}(x_2). \quad (2.3)$$

Lemma 2.2. (Hölder inequality [12]) *If θ_1 and θ_2 are positive real numbers such that $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$, then the following inequality holds for integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$:*

$$\left| \int_a^b f(x)g(x)dx \right| \leq \left(\int_a^b |f|^{\theta_1}dx \right)^{\frac{1}{\theta_1}} \left(\int_a^b |g|^{\theta_2}dx \right)^{\frac{1}{\theta_2}}. \quad (2.4)$$

Theorem 2.1. *If x, y, x_1, y_1 are positive real numbers satisfying the condition*

$$(x - x_1)(y - y_1) \geq 0, \quad (2.5)$$

then for the extended (p, q) -beta function, we have the inequality

$$\beta_{p,q}(x, y_1)\beta_{p,q}(x_1, y) \leq \beta_{p,q}(x_1, y_1)\beta_{p,q}(x, y), \quad (2.6)$$

Proof. Consider the mappings $f, g, h : [0, 1] \rightarrow [0, \infty)$ given by

$$f(t) = t^{x-x_1}, g(t) = (1-t)^{y-y_1} \text{ and } h(t) = t^{x_1-1}(1-t)^{y_1-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right).$$

Now, differentiation of f and g gives

$$f'(t) = (x - x_1)t^{x-x_1-1}, \quad g'(t) = (y - y_1)(1-t)^{y-y_1-1}.$$

This show that f and g have the same monotonicity on $[0, 1]$.

Applying the Chebyshev's integral inequality (2.1), for the above defined functions f, g and h , we have

$$\begin{aligned} & \left(\int_a^b t^{x-1}(1-t)^{y_1-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt \right) \left(\int_a^b t^{x_1-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt \right) \\ & \leq \left(\int_a^b t^{x_1-1}(1-t)^{y_1-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt \right) \left(\int_a^b t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt \right) \end{aligned}$$

which implies that,

$$\beta_{p,q}(x, y_1) \beta_{p,q}(x_1, y) \leq \beta_{p,q}(x_1, y_1) \beta_{p,q}(x, y)$$

which completes the desired proof. \square

Theorem 2.2. *The function $(p, q) \mapsto \beta_{p,q}(x, y)$ is log convex on $(0, \infty)$ for each $x, y > 0$. Moreover, the function $\beta_{p,q}(x, y)$ satisfy the following Turán type inequality*

$$\beta_{p,q}^2(x, y) - \beta_{p+a, q+a}(x, y) \beta_{p-a, q-a}(x, y) \leq 0, \quad (2.7)$$

for all real a .

Proof. From the definition of log-convexity, it will be sufficient to prove that

$$\beta_{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2}(x, y) \leq \left(\beta_{p,q}(x, y) \right)^\alpha \left(\beta_{p,q}(x, y) \right)^{1-\alpha}, \quad (2.8)$$

for $\alpha \in [0, 1]$, $p_1, p_2, q_1, q_2 > 0$ and for a fixed $x, y > 0$.

Obviously, (2.8) is true for $\alpha = 0$ and $\alpha = 1$. Assume that $\alpha \in (0, 1)$, then it follows from (1.5) that

$$\begin{aligned} & \beta_{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2}(x, y) \\ & = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{-\alpha p_1 - (1-\alpha)p_2}{t} + \frac{-\alpha q_1 - (1-\alpha)q_2}{1-t}\right) dt \\ & = \left(\int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p_1}{t} - \frac{q_1}{1-t}\right) dt \right)^\alpha \\ & \quad \times \left(\int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p_1}{t} - \frac{q_1}{1-t}\right) dt \right)^{1-\alpha} \end{aligned} \quad (2.9)$$

Let $\theta_1 = \frac{1}{\alpha}$ and $\theta_2 = \frac{1}{(1-\alpha)}$. Clearly $\theta_1 > 1$ and $\theta_1 + \theta_2 = \theta_1 \theta_2$. Thus applying the Hölder-Rogers inequality (2.4) for integrals in (2.9) gives

$$\beta_{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2}(x, y) < \left(\int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p_1}{t} - \frac{q_1}{1-t}\right) dt \right)^\alpha$$

$$\begin{aligned}
& \times \left(\int_0^1 t^{x-1} (1-t)^{y-1} \exp \left(-\frac{p_1}{t} - \frac{q_1}{1-t} \right) dt \right)^{1-\alpha} \\
& = \left(\beta_{p,q}(x, y) \right)^\alpha \left(\beta_{p,q}(x, y) \right)^{1-\alpha}.
\end{aligned} \tag{2.10}$$

This implies that $(p, q) \mapsto \beta_{p,q}(x, y)$ is log convex on $(0, \infty)$.

Now, taking $\alpha = \frac{1}{2}$, $p_1 = p - a$, $p_2 = p + a$, and $q_1 = p - a$, $q_2 = p + a$, the inequality (2.10) yields

$$\beta_{p,q}^2(x, y) - \beta_{p+a,q+a}(x, y) \beta_{p-a,q-a}(x, y) \leq 0.$$

□

Theorem 2.3. *The function $(x, y) \mapsto \beta_{p,q}(x, y)$ is logarithmic convex on $(0, \infty) \times (0, \infty)$, for all $p, q \geq 0$. In particular*

$$\beta_{p,q}^2 \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \leq \beta_{p,q}(x_1, y_1) \beta_{p,q}(x_2, y_2).$$

Proof. Let $(x_1, y_1), (x_2, y_2) \in (0, \infty)^2$, and $c, d \geq 0$ with $c + d = 1$, then we have

$$\beta_{p,q} \left(c(x_1, y_1) + d(x_2, y_2) \right) = \beta_{p,q}(cx_1 + dx_2, cy_1 + dy_2). \tag{2.11}$$

Applying the definition of (p, q) -extended beta function on the right hand side of inequality (2.11), we have

$$\begin{aligned}
& \beta_{p,q} \left(c(x_1, y_1) + d(x_2, y_2) \right) \\
& = \int_0^1 t^{cx_1+dx_2-1} (1-t)^{cy_1+dy_2-1} \exp \left(-\frac{p}{t} - \frac{q}{1-t} \right) dt \\
& = \int_0^1 t^{cx_1+dx_2-(c+d)} (1-t)^{cy_1+dy_2-(c+d)} \exp \left(-\frac{p(c+d)}{t} - \frac{q(c+d)}{1-t} \right) dt \\
& = \int_0^1 t^{c(x_1-1)} t^{d(x_2-1)} (1-t)^{c(y_1-1)} (1-t)^{d(y_2-1)} \exp \left(-\frac{pc}{t} - \frac{qc}{1-t} \right) \exp \left(-\frac{pd}{t} - \frac{qd}{1-t} \right) dt \\
& = \int_0^1 \left(t^{x_1-1} (1-t)^{y_1-1} \exp \left(-\frac{p}{t} - \frac{q}{1-t} \right) \right)^c \left(t^{x_2-1} (1-t)^{y_2-1} \exp \left(-\frac{p}{t} - \frac{q}{1-t} \right) \right)^d dt.
\end{aligned}$$

Again by considering $\theta_1 = \frac{1}{c}$, $\theta_2 = \frac{1}{d}$, we can use the Hölder-Rogers inequality for above integrals and it follows

$$\begin{aligned}
\beta_{p,q} \left(c(x_1, y_1) + d(x_2, y_2) \right) & \leq \left(\int_0^1 t^{x_1-1} (1-t)^{y_1-1} \exp \left(-\frac{p}{t} - \frac{q}{1-t} \right) dt \right)^c \\
& \times \left(\int_0^1 t^{x_2-1} (1-t)^{y_2-1} \exp \left(-\frac{p}{t} - \frac{q}{1-t} \right) dt \right)^d
\end{aligned}$$

$$= \left(\beta_{p,q}(x_1, y_1) \right)^c \left(\beta_{p,q}(x_2, y_2) \right)^d.$$

This shows the logarithmic convexity of extended (p, q) -beta function $\beta_{p,q}(x, y)$ on $(0, \infty)^2$. For $c = d = \frac{1}{2}$, the above inequality reduces to

$$\beta_{p,q}^2\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \leq \beta_{p,q}(x_1, y_1)\beta_{p,q}(x_2, y_2). \quad (2.12)$$

Let $x, y > 0$ be such that $\min_{a \in \mathbb{R}}(x + a, x - a) > 0$, then by taking $x_1 = x + a$, $x_2 = x - a$, $y_1 = y + b$ and $y_2 = y - b$ in (2.12), we get

$$\left[\beta_{p,q}(x, y) \right]^2 \leq \beta_{p,q}(x + a, y + b)\beta_{p,q}(x - a, y - b), \quad (2.13)$$

for all $p, q \geq 0$. □

3. INEQUALITIES FOR (p, q) -EXTENDED CONFLUENT HYPERGEOMETRIC FUNCTION

In this section, we present the log-convexity and Turán type inequality for extended confluent hypergeometric function defined in (1.6). For this continuation, we recall the following well-known lemma.

Lemma 3.1. [2] Consider the power series $f(x) = \sum_{n \geq 0} a_n x^n$ and $g(x) = \sum_{n \geq 0} b_n x^n$, where $a_n \in \mathbb{R}$ and $b_n > 0$ for all n . Further assume that both series converge on $|x| < \alpha$. If the sequence $\{a_n/b_n\}_n \geq 0$ is increasing (or decreasing), then $x \mapsto f(x)/g(x)$ is also increasing (or decreasing) function on $(0, \alpha)$.

Note that the above lemma is valid only if both f and g are both even or both odd functions.

Theorem 3.1. Let $\beta \geq 0$ and $\gamma, \delta > 0$, then the following assertions for extended (p, q) -confluent hypergeometric function are true.

- (i) For $\gamma \geq \delta$, the function $x \mapsto \Phi_{p,q}(\beta; \gamma; x)/\Phi_{p,q}(\beta; \delta; x)$ is increasing on $(0, \infty)$.
- (ii) For $\gamma \geq \delta$, we have $\delta \Phi_{p,q}(\beta + 1; \gamma + 1; x) \Phi_{p,q}(\beta; \delta; x) \geq \gamma \Phi_{p,q}(\beta; \gamma; x) \Phi_{p,q}(\beta + 1; \delta + 1; x)$.
- (iii) The function $x \mapsto \Phi_{p,q}(\beta; \gamma; x)$ is log-convex on \mathbb{R} .
- (iv) The function $(p, q) \mapsto \Phi_{p,q}(\beta; \gamma; x)$ is log convex on $(0, \infty)$ for fixed $x > 0$.
- (v) Let $\sigma > 0$. then the function

$$\beta \mapsto \frac{\beta(\beta, \gamma) \Phi_{p,q}(\beta + \sigma; \gamma; x)}{\beta(\beta + \sigma, \gamma) \Phi_{p,q}(\beta; \gamma; x)}$$

is decreasing on $(0, \infty)$ for fixed $\gamma, x > 0$.

Proof. From the definition of (1.6), it follows that

$$\frac{\Phi_{p,q}(\beta; \gamma; x)}{\Phi_{p,q}(\beta; \delta; x)} = \frac{\sum_{n=0}^{\infty} a_n(c) x^n}{\sum_{n=0}^{\infty} a_n(d) x^n}, \text{ where } a_n(x) = \frac{\beta_{p,q}(\beta + n, t - \beta)}{\beta_{p,q}(\beta, t - \beta)}. \quad (3.1)$$

If we denote $f_n = a_n(c)/a_n(d)$, then

$$\begin{aligned} f_n - f_{n+1} &= \frac{a_n(c)}{a_n(d)} - \frac{a_{n+1}(c)}{a_{n+1}(d)} \\ &= \frac{\beta(\beta, \delta - \beta)}{\beta(\beta, \gamma - \beta)} \left(\frac{\beta_{p,q}(\beta + n, \gamma - \beta)}{\beta_{p,q}(\beta + n, \delta - \beta)} - \frac{\beta_{p,q}(\beta + n + 1, \gamma - \beta)}{\beta_{p,q}(\beta + n + 1, \delta - \beta)} \right). \end{aligned}$$

Now take $x = \beta + n$, $y = \delta - \beta$, $x_1 = \beta + n + 1$, $y_1 = \gamma - \beta$ in (2.6). Since $(x - x_1)(y - y_1) = \gamma - \delta \geq 0$, it follows from Theorem 2.1 that

$$\frac{\beta_{p,q}(\beta + n, \gamma - \beta)}{\beta_{p,q}(\beta + n, \delta - \beta)} \leq \frac{\beta_{p,q}(\beta + n + 1, \gamma - \beta)}{\beta_{p,q}(\beta + n + 1, \delta - \beta)},$$

this is equivalent to say that $\{f_n\}$ is an increasing sequence and hence with the aid of Lemma 3.1, we observe that $x \mapsto \Phi_{p,q}(\beta; \gamma; x)/\Phi_{p,q}(\beta; \delta; x)$ is increasing on $(0, \infty)$.

To prove the assertion (ii), we recall the following well-known identity from [6]:

$$\frac{d^n}{dx^n} \Phi_{p,q}(\beta; \gamma; x) = \frac{(\beta)_n}{(\gamma)_n} \Phi_{p,q}(\beta + n; \gamma + n; x). \quad (3.2)$$

Since the increasing properties of $x \mapsto \Phi_{p,q}(\beta; \gamma; x)/\Phi_{p,q}(\beta; \delta; x)$ is equivalent to the following inequality

$$\frac{d}{dx} \left(\frac{\Phi_{p,q}(\beta; \gamma; x)}{\Phi_{p,q}(\beta; \delta; x)} \right) \geq 0. \quad (3.3)$$

This together with (3.2) implies

$$\begin{aligned} \Phi'_{p,q}(\beta; \gamma; x) \Phi_{p,q}(\beta; \delta; x) &- \Phi_{p,q}(\beta; \gamma; x) \Phi'_{p,q}(\beta; \delta; x) \\ &= \frac{\beta}{\gamma} \Phi_{p,q}(\beta + 1; \gamma + 1; x) \Phi_{p,q}(\beta; \delta; x) \\ &- \frac{\beta}{\delta} \Phi_{p,q}(\beta; \gamma; x) \Phi_{p,q}(\beta + 1; \delta + 1; x) \geq 0. \end{aligned}$$

This implies that

$$\delta \Phi_{p,q}(\beta + 1; \gamma + 1; x) \Phi_{p,q}(\beta; \delta; x) \geq \gamma \Phi_{p,q}(\beta; \gamma; x) \Phi_{p,q}(\beta + 1; \delta + 1; x)$$

which prove the assertion.

The log-convexity of $x \mapsto \Phi_{p,q}(\beta; \gamma; x)$ can be prove by using the integral representation of extended (p, q) -confluent hypergeometric function as given in (1.7) and by applying the Hölder-Rogers inequality for integrals as follows:

$$\begin{aligned} &\Phi_{p,q}(\beta; \gamma; \alpha x + (1 - \alpha)y) \\ &= \frac{1}{\beta(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1 - t)^{\gamma-\beta-1} \exp \left(\alpha x t + (1 - \alpha) y t - \frac{p}{t} - \frac{q}{1 - t} \right) dt \\ &= \frac{1}{\beta(\beta, \gamma - \beta)} \int_0^1 \left[\left(t^{\beta-1} (1 - t)^{\gamma-\beta-1} \exp \left(x t - \frac{p}{t} - \frac{q}{1 - t} \right) \right)^\alpha \right. \end{aligned}$$

$$\begin{aligned}
& \times \left(t^{\beta-1} (1-t)^{\gamma-\beta-1} \exp \left(yt - \frac{p}{t} - \frac{q}{1-t} \right) \right)^{1-\alpha} dt \\
& \leq \left[\frac{1}{\beta(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \exp \left(xt - \frac{p}{t} - \frac{q}{1-t} \right) dt \right]^\alpha \\
& \times \left[\frac{1}{\beta(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \exp \left(xt - \frac{p}{t} - \frac{q}{1-t} \right) dt \right]^{1-\alpha} \\
& = \left(\Phi_{p,q}(\beta; \gamma; x) \right)^\alpha \left(\Phi_{p,q}(\beta; \gamma; y) \right)^{1-\alpha}, (x, y > 0, \alpha \in [0, 1]).
\end{aligned}$$

This prove that $x \mapsto \Phi_{p,q}(\beta; \gamma; x)$ is log-convex for a fixed $x > 0$. For the case when $x < 0$, then the assertion immediately follows from the identity (see [6]):

$$\Phi_{p,q}(\beta; \gamma; x) = e^x \Phi_{q,p}(\gamma - \beta; \gamma; -x).$$

Since, the infinite sum of log-convex functions is log-convex for $x > 0$. Thus, the log-convexity of $(p, q) \mapsto \Phi_{p,q}(\beta; \gamma; x)$ is equivalent to prove that $(p, q) \mapsto \beta(\beta + n, \gamma - \beta)$ is log-convex on $(0, \infty)$ and for non-negative integer n . From Theorem 2.2, it is clear that $(p, q) \mapsto \beta(\beta + n, \gamma - \beta)$ is log-convex for $\gamma > \beta > 0$ and hence assertion (iv) is true.

Now, let $\beta' \geq \beta$ and set $h(t) = t^{\beta'-1} (1-t)^{\gamma-\beta'-1} \exp \left(xt - \frac{p}{t} - \frac{q}{1-t} \right)$, $f(t) = \left(\frac{t}{1-t} \right)^{\beta-\beta'}$ and $g(t) = \left(\frac{t}{1-t} \right)^\sigma$. Then using the integral representation (1.7) of extended confluent hypergeometric function, we have

$$\begin{aligned}
& \frac{\beta_{p,q}(\beta, \gamma) \Phi_{p,q}(\beta + \sigma; \gamma; x)}{\beta_{p,q}(\beta + \sigma, \gamma) \Phi_{p,q}(\beta; \gamma; x)} - \frac{\beta_{p,q}(\beta', \gamma) \Phi_{p,q}(\beta' + \sigma; \gamma; x)}{\beta_{p,q}(\beta' + \sigma, \gamma) \Phi_{p,q}(\beta'; \gamma; x)} \\
& = \frac{\int_0^1 f(t) g(t) h(t) dt}{\int_0^1 f(t) h(t) dt} - \frac{\int_0^1 g(t) h(t) dt}{\int_0^1 h(t) dt}.
\end{aligned} \tag{3.4}$$

One can easily determine that for $\beta' \geq \beta$, the function f is decreasing when $\sigma \geq 0$ and the function g is increasing. Since h is non negative function for $t \in [0, 1]$. Thus, by reverse Chebyshev's reverse inequality (2.1), it follows that

$$\int_0^1 f(t) h(t) dt \int_0^1 g(t) h(t) dt \leq \int_0^1 h(t) dt \int_0^1 f(t) g(t) h(t) dt. \tag{3.5}$$

This together with (3.4) implies

$$\frac{\beta_{p,q}(\beta, \gamma) \Phi_{p,q}(\beta + \sigma; \gamma; x)}{\beta_{p,q}(\beta + \sigma, \gamma) \Phi_{p,q}(\beta; \gamma; x)} - \frac{\beta_{p,q}(\beta', \gamma) \Phi_{p,q}(\beta' + \sigma; \gamma; x)}{\beta_{p,q}(\beta' + \sigma, \gamma) \Phi_{p,q}(\beta'; \gamma; x)} \geq 0,$$

which is equivalent to say that the function

$$\beta \mapsto \frac{\beta_{p,q}(\beta, \gamma) \Phi_{p,q}(\beta + \sigma; \gamma; x)}{\beta_{p,q}(\beta + \sigma, \gamma) \Phi_{p,q}(\beta; \gamma; x)}$$

is decreasing on $(0, \infty)$. □

Remark 3.1. *In particular, the following decreasing property of extended (p, q) -confluent hypergeometric function*

$$\beta \mapsto \frac{\beta_{p,q}(\beta, \gamma) \Phi_{p,q}(\beta + \sigma; \gamma; x)}{\beta_{p,q}(\beta + \sigma, \gamma) \Phi_{p,q}(\beta; \gamma; x)}$$

is equivalent to the following inequality

$$\Phi_{p,q}^2(\beta + \sigma; \gamma; x) \geq \frac{\beta_{p,q}^2(\beta + \sigma, \gamma)}{\beta_{p,q}(\beta + 2\sigma, \gamma) \beta(\beta, \gamma)} \Phi_{p,q}(\beta + 2\sigma; \gamma; x) \Phi_{p,q}(\beta; \gamma; x). \quad (3.6)$$

When $p = q$, then the above inequality will reduce to the inequality recently proved by [11]. Similarly, when $p = q = 0$, then the above inequality reduces to the inequality of confluent hypergeometric which is an improved version of Theorem 4(b) given in [9].

4. CONCLUSION

In this paper, we introduced inequalities for extended (p, q) -beta and (p, q) -confluent hypergeometric function defined by Choi et al. [6]. Throughout in this paper, if we take $p = q$ then we get the inequalities of extended beta function and extended confluent hypergeometric function recently introduced by Mondal [11]. Similarly if we take $p = q = 0$, then the newly defined inequalities for extended (p, q) -beta function will reduce to the inequalities of classical beta function (see [1, 7]).

REFERENCES

- [1] R. P. Agarwal, N. Elezović, J. Pečarić, *On some inequalities for beta and gamma functions via some classical inequalities*, J. Inequal. Appl., no. 5 (2005), 593-613.
- [2] M. Biernacki, J. Krzyz, *On the monotonicity of certain fractionals in the theory of analytic functions*, Ann. Univ. Mariae Curie-Sklodowska. Sect. A. 9 (1955), pp. 135-147.
- [3] S. Butt, J. Pecaric, A. Rehman, *Exponential convexity of Petrovic and related functional*, J. Inequal. Appl. 2011, 89 (2011).
- [4] M. A. Chaudhry, S. M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math. 55, 99124, (1994).
- [5] M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, *Extension of Eulers beta function*, J. Comput. Appl. Math. 78 (1997) 1932.
- [6] J. Choi, A. K. Rathie, R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Mathematical J. 36 (2014), No. 2, pp. 357-385.
- [7] S. S. Dragomir, R. P. Agarwal, N. S. Barnett, *Inequalities for beta and gamma functions via some classical and new inequalities*, J. Inequal. Appl. 5, no. 2 (2000), 103-165.
- [8] P. Kumar, S. P. Singh, S. S. Dragomir, *Some inequalities involving beta and gamma functions*, Nonlinear Anal. Forum 6(1), 143-150 (2001).

- [9] D. Karp, S. M. Sitnik, *Log-convexity and log-concavity of hypergeometric-like function*, J. Math. Anal. Appl. 364, no. 2 (2010), 384-394.
- [10] D. S. Mitrinovic, J. E. Pecaric, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht (1993).
- [11] S. R. Mondal, *Inequalities of extended beta and extended hypergeometric functions*, J. Inequal. Appl. (2017) 2017:10
- [12] W. Rudin, *Real and Complex Analysis*, 3rd edn. McGraw-Hill International Editions (1987).
- [13] P. Turán, On the zeros of the polynomials of Legendre, Casopis pro Pestovani Mat. a Fys, 75 (1950), 113-122.

SHAHID MUBEEN: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SARGODHA, SARGODHA, PAKISTAN

E-mail address: smjhanda@gmail.com

KOTTAKKARAN SOOPPY NISAR: DEPARTMENT OF MATHEMATICS, COLLEGE OF ARTS AND SCIENCE-WADI AL DAWSER, 11991, PRINCE SATTAM BIN ABDULAZIZ UNIVERSITY, SAUDI ARABIA

E-mail address: ksnisar1@gmail.com, n.sooppy@psau.edu.sa

GAUHAR RAHMAN: DEPARTMENT OF MATHEMATICS, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD, PAKISTAN

E-mail address: gauhar55uom@gmail.com

MUHAMMAD ARSHAD: DEPARTMENT OF MATHEMATICS, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD, PAKISTAN

E-mail address: marshad_zia@yahoo.com